UDC 531.36

ON COMPLEX STRUCTURES OF A NONAUTONOMOUS PERIODIC PIECEWISE-LINEAR SYSTEM OF A CYLINDER

PMM Vol. 41, № 1, 1977, pp. 173-178
V. N. BELYKH and Iu. S. CHERTKOV
(Gor'kii)
(Received April 24, 1975)

A nonautonomous periodic piecewise-linear system of second order equations in the theory of phase synchronization is considered. Boundaries of the region of existence of a rough homoclinal curve are determined in an explicit form. The nature of the complex structure change when moving through these boundaries is indicated.

In the investigation of specific dynamic systems that are higher than two-dimensional, one of the difficult problems is that of analyzing systems of complex structure such as systems with a denumerable set of periodic motions. Interest in such systems is in many respects associated with the problem of stochastic motions of dynamic systems [1]. Methods of complex structure determination which were used in [1 — 14] are as follows: (a) direct analysis of solutions [9, 10] including numerical methods [11, 12]; (b) the topological method in dissipative systems [13]; (c) methods of symbolic dynamics in conservative systems [14]; (d) reduction to circle mapping [1], and (e) the determination of the rough (as well as nonrough) homoclinal curve [2 — 8].

1. Statement of the problem. Basic results. We consider the system of equations of the theory of phase synchronization [6] of the form (1.1) specified in a nonautonomous cylindrical phase space G in the parameter region D

$$\varphi^{\cdot} = y, \; y^{\cdot} = \gamma - (\lambda + aF'(\varphi))y - F(\varphi) + bh(t) \tag{1.1}$$

$$h(t) = \sin \omega t, \; F(\varphi) = 1 - \varphi / \pi, \; \varphi \pmod{2\pi} \tag{1.2}$$

$$G = \left\{\varphi \pmod{2\pi}, \; t \left(\bmod \frac{2\pi}{\omega}\right), \; y\right\}$$

$$D = \{\lambda \geqslant 0, \; a \geqslant 0, \; b \geqslant 0, \; \omega > 0, \; \gamma\}$$

Particular attention is given to the question of existence of complex structure; the existence and bifurcation of the denumerable number of periodic motions of the (p, q)-type, i.e. of periodic motions contained in G during time $p\tau$ after a number of revolutions $q > 0$ with respect to φ (p, q are integers) is considered.

We determine the region d_+ of parameters of system (1.1) that cprresponds to the existence of the homoclinal curve which according to [15, 16] guarantees the presence of structure

$$d_+\colon \; |\gamma - \gamma_0(\lambda, a)| < \varepsilon \tag{1.3}$$

$$\varepsilon = \frac{b}{\pi \sqrt{\Delta^2 + \omega^2\sigma^2}}, \quad \sigma = -\lambda + \frac{a}{\pi}, \quad \Delta = \omega^2 + \frac{1}{\pi}$$

$$\gamma_0(\lambda, a) = \frac{\pi\lambda + a}{\sqrt{(\pi\lambda - a)^2 + 4\pi}} \tag{1.4}$$

When in the autonomous system (1.1) $b = 0$, function $\gamma = \gamma_0(\lambda, a)$ corresponds to the bifurcation of the separatrix loop of that system saddle, which envelops the upper semicylinder.

In region d_+ the complex structure is determined by the denumerable set of groups of periodic motions of the (p, q)-type ($q = 1, 2, \ldots$), each group consists of a denumerable number of periodic motions (p^*, q) of the $(p^* + 1, q)$-types. The complex structure does not vanish at exit from region d_+ with the change of parameters across the boundary γ_+ (when $\sigma < 0$) and γ_- (when $\sigma > 0$), where $\gamma_+ = \gamma_0(\lambda, a) + \varepsilon$ and $\gamma_- = \gamma_0(\lambda, a) - \varepsilon$, but changes at the boundary, and is then determined by the denumerable set of groups of (p, q)-type periodic motions ($q = 1, 2, \ldots$). Each of these consists by then of a finite number of periodic motions of the $(p_1, q), (p_2, q), \ldots, (p_k, q)$-types.

In the language of symbolic sequencies generated by two images T and L on the local and global pieces of the extended region [16] of the homoclinal curve this has the following meaning. When the homoclinal curve of periodic sequences $\ldots T^{j_1} L T^{j_2} \ldots L T^{j_k} L T^{j_1} \ldots$ vanishes with one and the same number of operators L, the period gets a finite number ($j_1, j_2, \ldots, j_k$ are finite). The denumerability of periodic sequencies is obtained owing to the denumerability of the number of periods with a different number L in each period.

Farther away from the boundary γ_+ ($\sigma < 0$) or γ_- ($\sigma > 0$) the complex structure vanishes. At the exit from region d_+ through another part of boundaries γ_+ ($\sigma > 0$) and γ_- ($\sigma < 0$) the complex structure also vanishes (at the reserve passage an " Ω-explosion" takes place). The existence of complex structures and their bifurcation was obtained here with the use of methods (a) and (e) described above.

System (1.1) contains in addition to the described bifurcations also others similar to those obtained in [6, 7] which determine the change of the complex structure with periodic motions of the $(p, -q)$-type (reverse rotation with respect to φ) and the $(p, 0)$-type (oscillatory).

Finally, an estimate is given of the capture region of the phase synchronization system which corresponds to the global stability of system (1.1).

2. Supplementary definition of system (1.1). For solving this system we supplement its definition at discontinuity points of function $F(\varphi)$ by passing to limit $\nu \to 0$

$$F(\varphi) = \begin{cases} \varphi/\nu, & \varphi \in (-\nu, \nu], \\ (\pi - \varphi)/(\pi - \nu), & \varphi \in (\nu, 2\pi - \nu], \end{cases} \qquad 0 < \nu < \pi \tag{2.1}$$

We assume that the forced periodic saddle motion of system (1.1), (2.1) does not extend beyond the boundaries of region $(\nu, 2\pi - \nu)$ for $\nu \in (0, \pi)$. For system (1.1) this implies that the limit inequality

$$\varepsilon \leqslant \min\{1 - \gamma, 1 + \gamma\} \tag{2.2}$$

is satisfied.

As the result of passing to limit of $\nu \to 0$, we obtain that along section $(0, 2\pi]$ solutions of system (1.1), (1.2) are of the form

$$\varphi = C_1 e^{s_1 t} + C_2 e^{s_2 t} + A \sin \omega t + B \cos \omega t + \pi(1 - \gamma), \qquad y = \varphi \tag{2.3}$$

where

$$s_{1,2} = \frac{1}{2}(\sigma \pm \sqrt{\sigma^2 + 4/\pi}), \quad A = -\frac{b\Delta}{\Delta^2 + \omega^2\sigma^2}, \quad B = \frac{\omega b \sigma}{\Delta^2 + \omega^2\sigma^2} \tag{2.4}$$

The joining of points $\varphi = 0$ and $\varphi = 2\pi$ is carried out as follows:

1) if $\varphi(t_1-0, t_0) = 2\pi - 0$ and $y(t_1-0, t_0) = y_1 > 2a$, the solution is extended so that $\varphi(t_1, t_1) = +0$ and $y(t_1, t_1) = y(t_1-0, t_0) - 2a$;

2) if $\varphi(t_1-0, t_0) = +0$ and $y(t_1-0, t_0) < -2a$, the solution is extended so that $\varphi(t_1, t_1) = 2\pi - 0$ and $y(t_1, t_1) = y(t_1-0, t_0) + 2a$;

3) if $\varphi(t_1-0, t_0) = 2\pi - 0(+0)$ and $|y(t_1-0, t_0)| \leqslant 2a$, the solution is extended by motions of the form $\varphi = 0$ and $y = g(t)$ which within the finite time t^* approach the limit motion $\varphi \equiv 0$ and $y \equiv 0$ that represents a steady periodic motion.

3. The region of existence of rough homoclinal curve. According to Poincaré's definition a homoclinal curve is a doubly asymptotic trajectory to a periodic saddle motion. The homoclinal curve is obviously the intersection of separatrix manifolds of stable and unstable periodic saddle motion. If such intersection is transversal, the homoclinal curve is called rough [15, 16] and in the opposite case nonrough [19].

Let us determine the conditions of existence of a rough homoclinal curve of the periodic saddle motion of system (1.1), (1.2) in the region $\varphi \in (0, 2\pi)$ of the form (2.3) $(C_1 = C_2 = 0)$

$$\varphi^* = A \sin \omega t + B \cos \omega t + \pi(1-\gamma), \quad y^* = \frac{d\varphi^*}{dt} \tag{3.1}$$

Equations of the stable and unstable separatrix surfaces W^+ and W^- are of the form

$$W^+:\ y = y^*(t) + s_2(\varphi - \varphi^*(t) - 2\pi) \tag{3.2}$$

$$W^-:\ y = y^*(t) + s_1(\varphi - \varphi^*(t))$$

(s_1 corresponds to the plus sign in (2.4)). Manifolds W^+ and W^- are extended at transition through $\varphi = 2\pi$ by solutions (2.3) in conformity with supplemental procedure described above.

Let l^+ and l^- be the curves of intersection of surfaces W^+ and W^- with the plane $\varphi = 2\pi + 0$ in the phase space G. The relative position of surface W^+ and W^- relative to each other is defined by the relative position of curves l^+ and l^-. Intersection points of curves l^+ and l^- correspond to the homoclinal curve of system (1.1). Let us denote the equations of the curves l^+ and l^- by $y^+(t)$ and $y^-(t)$, respectively. The condition of existence of a simple root of equation

$$y^+(t) - y^-(t) = 0 \tag{3.3}$$

is equivalent to the condition of existence of a rough homoclinal curve. The substitution of $\varphi = 2\pi$ into the first and second of Eqs. (3.2) yields, respectively, functions $y^+(t)$, and $y^-(t)$ in the region of parameters d_1

$$\left|\pi(1+\gamma) - \frac{b\sqrt{\omega^2 + s_1^2}}{\sqrt{\Delta^2 + \omega^2\sigma^2}}\right| > 2a \tag{3.4}$$

which satisfy the inequality $\bar{y}^-(t) > 2a$. Then, assuming that condition (3.4) is satisfied below, from the supplemental process we obtain $y^-(t) = y^-(t) - 2a$. As the result (3.3) is transformed into an equation of the form

$$\sin(\omega t - \alpha) = \frac{\pi}{b}\sqrt{\Delta^2 + \omega^2\sigma^2}\left(\gamma - \frac{\pi\lambda + a}{\sqrt{\pi^2\sigma^2 + 4\pi}}\right), \quad \alpha = \operatorname{arctg}\frac{\sigma\omega}{\Delta} \tag{3.5}$$

which has two simple roots in a period when inequalities (1.3) are satisfied. Using the Neimark-Shil'nikov theorem [15, 16] (the results of [15, 16] and also [19] are readily transferable to the case of joined systems [20]) we obtain the following statement.

Lemma 1. In the parameter region d_+ system (1.1) contains a denumerable number of sets ($q = 1, 2, \ldots$) each of which consists of a denumerable number of periodic motions of the (p, q)-type (q is fixed, $p = 1, 2, \ldots$), which means that it has a complex structure.

At the boundary of region d_+ ((1.3) becomes an equality) the tangency of manifolds W^+ and W^- is of the quadratic kind. Furthermore the product of multipliers of the periodic saddle motion [19] satisfies the inequalities $\exp(s_1 + s_2)\tau - 1 < 0 (> 0)$ when $\sigma < 0$ ($\sigma > 0$).

Thus the conditions of the Gavrilov-Shil'nikov theorem [19], according to which the complex structure vanishes at transition through boundaries $\gamma = \gamma_-$ ($\sigma < 0$) and $\gamma = \gamma_+$ ($\sigma > 0$) and is maintained at transition through boundaries $\gamma = \gamma_-$ ($\sigma > 0$) and $\gamma = \gamma_+$ ($\sigma < 0$) are satisfied, although in that case the intersection of W^+ and W^- vanishes. Hence the following statement is valid.

Lemma 2. There exists a μ dependent on parameters such that system (1.1) has a complex structure in region

$$\gamma_+ + \mu > \gamma \geqslant \gamma_+ \ (\sigma < 0), \quad \gamma_- - \mu < \gamma \leqslant \gamma_- \ (\sigma > 0) \tag{3.6}$$

4. Changes of the complex structure. Let us elucidate the changes which the complex structure undergoes at transition through the boundaries $\gamma = \gamma_-$ ($\sigma > 0$) and $\gamma = \gamma_+$ ($\sigma < 0$). For this we shall prove the following lemma.

Lemma 3. System (1.1) can have a denumerable number of periodic motions (p, q) of the fixed-q type only in the region of existence of the homoclinal curve.

Proof. We denote solutions (2.3) in which $C_{1,2}$ are expressed in terms of input conditions by formulas

$$C_i = \frac{(-1)^i}{s_2 - s_1}\{(y_0 - A\omega\cos\omega t_0 + B\omega\sin\omega t_0) + (\pi s_i)^{-1}[\varphi_0 - A\sin\omega t_0 - B\cos\omega t_0 - \pi(1-\gamma)]\}, \quad i = 1, 2 \tag{4.1}$$

by $\varphi = \Phi(t, t_0, \varphi_0, y_0)$ and $y = Y(t, t_0, \varphi_0, y_0)$

The conditions of existence of periodic motions of the (p, q)-type are of the form

$$\Phi(t_{i+1}, t_i, 0, y_i - 2a\,\mathrm{sign}\,i) = 2\pi \qquad (\mathrm{sign}\,0 = 0) \tag{4.2}$$

$$Y(t_{i+1}, t_i, 0, y_i - 2a\,\mathrm{sign}\,i) = y_{i+1}$$

$$i = 0, 1, \ldots, q-1, \quad t_q = t_0 + p\tau\left(\tau = \frac{2\pi}{\omega}\right), \quad y_q = y_0 + 2a$$

System (4.2) consists of $2q$ equations in $2q$ unknown t_i and y_i ($i = 0, 1, \ldots, q-1$). The form of functions in (4.2) (see (2.3) and (4.1)) implies that that system can have a denumerable number of solutions only when it has at least one solution, if only for one $\Delta t_i \to \infty$ ($i > 0$). Setting in (4.2), for example, $\Delta t_1 \to \infty$, we obtain the system of equations of the form

$$\varepsilon\sin\left(\omega t_0 - \alpha + \frac{\omega}{s_1}\ln Q_{21}\right) = 1 + \gamma - \frac{2(s_2 - a/\pi)}{s_2 - s_1}\sum_{j=1}^{q} Q_{j1} \tag{4.3}$$

$$\varepsilon\sin\left(\omega t_0 - \alpha + \frac{\omega}{s_1}\ln Q_{qi}\right) = \gamma - 1 + \frac{2(s_1 - a/\pi)}{s_1 - s_2}\sum_{j=1}^{q} Q_{qj}^{\varkappa}$$

$$Q_{ij} = \lim_{\Delta t_1 \to \infty}\exp[-s_1(t_i - t_j)]; \quad \varkappa = \left|\frac{s_2}{s_1}\right|, \quad \Delta t_i = t_i - t_{i-1}$$

(If other $\Delta t_i \to \infty$, the corresponding Q_{ij} vanish).

Taking into account the boundedness of the left-hand side of (4.3), we obtain in the case of $a\lambda < 1$ the necessary conditions of solution (4.3) $|\gamma - \gamma_0(\lambda, a)| < \varepsilon$ which is the same as the condition of existence of a rough homoclinal curve (1.3). The lemma is proved.

From Lemmas 2 and 3 follows the theorem.

Theorem. In the parameter region (3.6) system (1.1) has a complex structure that is formed by the denumerable number of sets of periodic motions of the (p, q)-type $(q = 1, 2, \ldots)$ each of which consists of a finite number of periodic motions (q is fixed and p is finite).

Vanishing of the complex structure at exit from region (3.6) with the change of γ is associated with the vanishing of solutions of the system of Eqs. (4.3) of an infinite (denumerable) order when $q \to \infty$.

As an example, we consider the variation of the number of periodic solutions of the $(p, 1)$-type produced by the variation of parameters.

If $q = 1$ system (4.2) after elimination of y_0 assumes the form of equation

$$\varepsilon \sin(\omega t_0 - \alpha) = \gamma + \frac{(\lambda + a/\pi)(e^{s_1 \tau p} - e^{s_2 \tau p}) - (s_1 + s_2)(e^{(s_1+s_2)\tau p} - 1)}{(s_1 - s_2)(e^{s_1 \tau p} - 1)(e^{s_2 \tau p} - 1)} \quad (4.4)$$

Passing in (4.4) to limit $p \to \infty$, which means that $t_1 = t_0 + p\tau \to \infty$, and obtaining the roots of the derived equation, we find that the number of periodic motions of the $(p, 1)$-type of system (1.1) is denumerable in region d_+ (see (1.3)), finite in region $1 \geqslant \gamma > \gamma_+$ $(a < \pi\lambda)$, $\gamma < \gamma_-$ $(a > \pi\lambda)$, and that in region $\gamma \leqslant \gamma_-$ $(a < \pi\lambda)$, $1 \geqslant \gamma \geqslant \gamma_+$ $(a > \pi\lambda)$ the system has no such motions.

5. Other structures. The invariance of system (1.1) with respect to the substitutions $\varphi \to -\varphi$, $y \to -y$, $\gamma \to -\gamma$, $t \to t + \pi/\omega$ results in a symmetric subdivision of the parameter space relative to $\gamma = 0$. In particular, region d_- of the form d_-: $-\gamma_+(\lambda, a) < \gamma < -\gamma_-(\lambda, a)$ which is symmetric to d_+ and corresponds to the complex structure formed by the denumerable set of periodic motions of the $(p, -q)$-type. Since for small a region d_+ (d_-) encroaches on region $\gamma < 0$ (>0), there exists region $d_0 = d_+ \cap d_-$ which is determined by the inequalities d_0: $-\gamma_- > \gamma > \gamma_-$, at whose points the upper and lower pairs of separatrix manifolds intersect. Hence in region d_0 each of the two unstable intersect each of the two stable separatrix surfaces that constitute W^- and W^+. This implies that a complex structure formed by the denumerable set of periodic motions of the (p, q), $(p, -q)$ and $(p, 0)$-type corresponds to region d_0

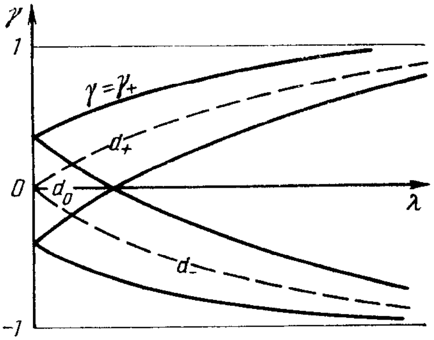

Fig. 1

Regions d_+, d_- and d_0 are shown in Fig. 1 in the plane (λ, γ) for $a = 0$. Cross sections $t = \text{const}$ of the manifolds W^- and W^+ which correspond to these regions appear in Fig. 2.

The estimates of the parameter regions given in [6], and also the symmetry of system

(1.1) with respect to substitutions $y \to -y$, $t \to -t - \pi/\omega$, $\lambda \to -\lambda$, $a \to -a$, yields a fairly complete picture of subdivision of the parameter space. Below we present only the information related to the region of global stability of system (1.1), which in the theory of phase synchronization is called [5] the capture region.

6. The capture region. In the autonomous case of $b = 0$ the capture region d_a of system (1.1) is determined for $\sigma < 0$ by the bifurcation of curve $\gamma = \gamma_0(\lambda, a)$ (see (1.3)) that corresponds to the existence of separatrix loop which envelops the upper semicylinder, and when $\sigma > 0$ by the bifurcation of curve $\gamma = \gamma^*(\lambda, a)$ that corresponds to the existence of a binary cycle which satisfies the condition $\gamma^*(\lambda, a) = \gamma_0(\lambda, a)$ for $\sigma = 0$ so that $d_a = \{|\gamma| < \gamma_0\ (\sigma < 0),\ |\gamma| < \gamma^*\ (\sigma > 0)\}$. (Function $\gamma^*(\lambda, a)$ is the solution of the system of transcendental equations presented in [21, 22]).

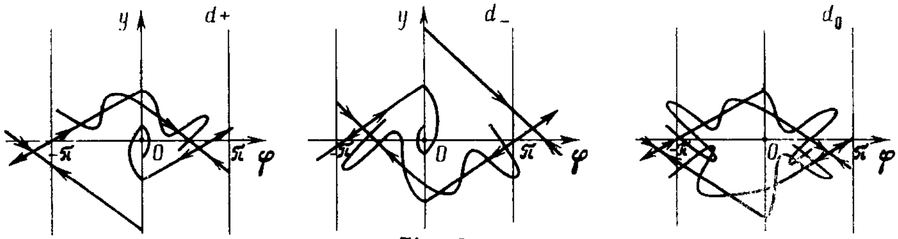

Fig. 2

According to the above investigation the capture region of the nonautonomous system (1.1) for which the solution $\varphi = 0$ at the joint attracts trajectories of the system except those lying on W^+, is determined for $\sigma < 0$ by the curve γ_-, i.e. $d_n = \{|\gamma| < \gamma_-\ (\sigma < 0)\}$, and for $\sigma > 0$, satisfies according to [6] the estimate $d_n \supset d^* = \{|\gamma| < \gamma^* - b\}$ The estimate of the parameter region in [6] for whose points system (1.1) contains at least one periodic motion of the (p, q)-type $(q > 0)$ when $\sigma > 0$ is of the form $\gamma > \gamma^* + b$. Hence the exact boundary of the capture region $\gamma = \gamma_L$ satisfies for $\sigma > 0$ the inequalities

$$\gamma_L - b < \gamma_L < \min\{\gamma^* + b, \gamma_-\}$$

and corresponds to bifurcation of the origination of periodic motions of the (p, q)-type similar to the bifurcation of the binary cycle in the autonomous case $b = 0$. The region of existence of the homoclinal curve d_+ is divided by curve $\sigma = 0$ into two parts. One of these, $d_+\ (\sigma < 0)$ borders on the capture region and, according to [5], is a region of quasi-capture, while the second $d_+\ (\sigma > 0)$ is separated from region d_n by the region lying between the bifurcation curves γ_- and γ_L and, consequently, is not a region of quasi-capture.

REFERENCES

1. Zaslavskii, G. M., Statistical Irreversibility in Nonlinear Systems. Moscow, "Nauka", 1973.
2. Mel'nikov, V. K., On the stability of the center at time-periodic perturbations. Tr. Mosk. Mathem. Obshch., Vol. 12, 1963.
3. Cherry, T. M., Asymptotic solutions of analytic Hamiltonian systems. L. Diff. Equat., Vol. 4, № 2, 1968.
4. Shilnikov, L. P., On the works of A. G. Maiem on central motions. Matem. Zametki, Vol. 5, № 3. 1969.
5. Beliustina, L. N. and Belykh, V. N., On the nonautonomous phase system of equations with a small parameter, containing invariant anchor rings and rough homoclinal curves. Izv, VUZ, Radiofizika, Vol. 15, № 7, 1972.

6. Beliustina, L. N. and Belykh, V. N., On the global structure of phase space subdivision of a particular nonautonomous system. Differentsial'nye Uravneniia, Vol. 9, № 4, 1973.
7. Beliustina, L. N. and Belykh, V. N., Homoclinal structures generated by the simplest model of phase automatic control. Coll.: Phase Synchronization, Moscow, "Sviaz", 1975.
8. Neimark, Iu. I., On the origin of stochastic properties in dynamic systems. Izv. VUZ, Radiofizika, Vol. 17, № 4, 1974.
9. Cartwright, M. L. and Littlewood, J. E., On nonlinear differential equations of the second order: the equation $y'' - k(1 - y^2)y' + y = b\lambda k \cos(\lambda t + \alpha)$, k large. J. London Math. Soc., Vol. 20, pt. 2, № 78, 1945.
10. Levinson, N., A second order differential equation with singular solutions. Ann. Math., Ser. 1, Vol. 50, № 1, 1949.
11. Hayashi, C., Ueda, Y. and Kawakami, H., Periodic solutions of Duffings equation with reference to doubly asymptotic solutions. Tr. Mezhdunar. Konferentsii po Nelineinym Kolebaniiam. Vol. 2, Kiev, Izd, Inst, Matematiki Akad. Nauk UkrSSR, 1970.
12. Batalova, Z. S. and Neimark, Iu. I., On a particular dynamic system of homoclinal structure. Theory of Oscillations, Applied Mathematics and Cybernetics. Mezhvuzovskii sb., № 1, Izd. Gor'kovsk. Univ., 1973.
13. Pliss, V. A., On the theory of invariant sets in periodic systems of differential equations. Differentsial'nye Uravneniia, Vol. 5, № 2, 1969.
14. Alekseev, V. M., Quasi-stochastic oscillations and qualitative problems of celestial mechanics. 9-year Mathematical college (Katsiveli, 1971). Kiev, Izd. Inst. Matematiki Akad. Nauk UkrSSR, 1972.
15. Shil'nikov, L. P., On a particular Poincaré-Birkhoff problem. Matem. sb., Vol. 74, № 3, 1967.
16. Neimark, Iu. I., Structure of dynamic systems motion in the neighborhood of homoclinal curve. 5-year Mathematical College (Uzhgorod, 1967), Izd. Akad. Nauk UkrSSR, Kiev, 1968.
17. Skriabin, B. N., Qualitative investigation of an equation of the theory of phase automatic frequency control. PMM Vol. 33, № 2, 1969.
18. Belykh, V. N., Kiveleva, K. G. and Fraiman, L. A., Dynamic characteristics of the scanning phase automatic frequency control system with first order filter. In coll.: Phase Synchronization, Moscow, "Sviaz'", 1975.
19. Gavrilov, N. K. and Shil'nikov, L. P., On three-dimensional dynamic systems with nonrough homoclinal curve, I, II. Matem. sb. Vol. 88, № 8, 1972 and Vol. 90, № 1, 1973.
20. Morozov, A. D., On piecewise-smooth systems containing homoclinal curves. Tr. Mezhdunar. Konferentsii po Nelineinym Kolebaniiam. Vol. 2, (Kiev, 1969). Kiev, Izd. Inst. Matematiki Akad. Nauk UkrSSR, 1970.
21. Shakhtarin, B. I., Investigation of a piecewise-linear system of the phase automatic control. Radiotekhnika i Elektronika, Vol. 14, № 8, 1969.
22. Safonov, V. M., On the effect of the sawtooth form of the phase detector characteristic on the phase automatic control capture band. Radiotekhnika, Vol. 24, № 6, 1969.

Translated by J. J. D.